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Generalized Moments of Additive Functions

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1. INTRODUCTION

Let f be a complex-valued additive arithmetical function. In this paper we want to give a characterization of the additive functions f, for which

$$\sup_{x\geqslant 1} x^{-1} \sum_{n\leqslant x} \phi(|f(n)|)<\infty, \tag{1}$$

where ϕ is a quite general nonnegative-valued increasing function. The case $\phi(x)=x^{\beta}$ was handled by Elliott [1] ($\beta>1$), Hildebrand and Spilker [4] ($\beta\geqslant 1$), and Indlekofer [6] ($\beta>0$). More general results were proved in Indlekofer [7], where ϕ satisfies the condition

$$\phi(x+y)\leqslant c_1(\phi(x)+\phi(y)) \qquad (x, y\geqslant 0) \tag{2}$$

or

$$\phi(x)=c_2^x \qquad (x\geqslant 0). \tag{3}$$

with positive constants c_1, c_2.

In the first case $\phi(x)$ does not increase higher than a power of x as x goes to infinity. Here we close the gap between (2) and (3) and assume that

$$\phi(x+y)\leqslant c_3\phi(x)\,\phi(y) \qquad \text{for all} \quad x, y\geqslant 1 \tag{4}$$

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(see Theorem 2 and the corollary). Further, we show that the assumption (4) is best possible in a certain sense (see Theorem 3).

2. Results

Here ϕ always denotes a nondecreasing function ϕ; $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $\lim_{x \to \infty} \phi(x) = \infty$. The class of functions satisfying (1) is denoted by L_ϕ. Then we prove

THEOREM 1. *Let f: $\mathbb{N} \to \mathbb{C}$ be additive, and let ϕ satisfy the inequality*

$$\phi(x-1) \gg \phi(x) \qquad \text{for} \quad x \geqslant 2.$$

Then, if $f \in L_\phi$, the series

$$\sum_{\substack{p \\ |f(p)| > 1}} \frac{1}{p}, \qquad \sum_{\substack{p \\ |f(p)| \leqslant 1}} \frac{|f(p)|^2}{p}, \qquad \sum_{\substack{p \\ |f(p^m)| > 1}} \sum_{m=1}^{\infty} \frac{\phi(|f(p^m)|)}{p^m} \tag{5}$$

converge and

$$\sum_{\substack{p \leqslant x \\ |f(p)| \leqslant 1}} \frac{f(p)}{p} = O(1) \qquad \text{as} \quad x \to \infty. \tag{6}$$

In the other direction we have

THEOREM 2. *Let f: $\mathbb{N} \to \mathbb{C}$ be additive, and suppose that*

$$\phi(x+y) \leqslant c_2 \phi(x)\,\phi(y) \qquad \text{for all} \quad x, y \in [1, \infty). \tag{7}$$

Then the convergence of the series (5) and the inequality (6) together imply $f \in \mathscr{L}_\phi$.

Theorems 1 and 2 give

COROLLARY. *Let f: $\mathbb{N} \to \mathbb{C}$ be additive and let ϕ satisfy the conditions of Theorem 2. Then $f \in \mathscr{L}_\phi$ if and only if the series (5) converge and (6) holds.*

Theorem 2 and the corollary are sharp in the following sense.

THEOREM 3. *Suppose that*

$$\sup_{x, y \geqslant 1} \frac{\phi(x+y)}{\phi(x)\,\phi(y)} = \infty. \tag{8}$$

Then there exists an additive function f such that the series in (5) *and* (6) *converge but $f \notin \mathcal{L}_\phi$.*

3. Notations and Lemmata

If f is an additive function then we define two functions f_1 and f_2 by

$$\begin{aligned} f_1(p^m) &= \begin{cases} f(p) & \text{if } \ m=1 \text{ and } |f(p)| \leqslant 1 \\ 0 & \text{otherwise} \end{cases} \\ f_2(p^m) &= \begin{cases} f(p) & \text{if } \ m=1 \text{ and } |f(p)| > 1 \\ f(p^m) & \text{if } \ m \geqslant 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

Obviously $f = f_1 + f_2$.

Lemma 1. *Let $f\colon \mathbb{N} \to \mathbb{R}$ be additive. Assume that for each $x \geqslant x_0$ there is a sequence of positive integers $a_1 < a_2 < \cdots < a_k \leqslant x$ with $k \geqslant c_1 x$, so that $|f(a_i)| \leqslant c_2$ $(i = 1, ..., k)$. Then the series*

$$\sum_{\substack{p \\ |f(p)| > 1}} \frac{1}{p} \quad \textit{and} \quad \sum_{\substack{p \\ |f(p)| \leqslant 1}} \frac{|f(p)|^2}{p}$$

converge, and, as $x \to \infty$,

$$\sum_{\substack{p \leqslant x \\ |f(p)| \leqslant 1}} \frac{f(p)}{p} = O(1).$$

The assertions of the lemma are easy consequences of a result of Erdős [3]. (See, for example, Elliott [2, Lemma 7.8; a similar proof is implicitly contained in Indlekofer [5, p. 261].)

Lemma 2. *Let $g\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be nonnegative and multiplicative with $g(p^m) \leqslant c$ for all primes p and all $m \in \mathbb{N}$. Then*

$$\sum_{n \leqslant x} g(n) \ll x \exp\left(\sum_{p \leqslant x} \frac{g(p) - 1}{p} \right). \tag{10}$$

(*The constant in $\ll$ depends only on c.*)

This lemma can be proved by a well-known method of Wirsing (see, for example, Indlekofer [6, Lemma 2]).

4. Proof of Theorem 1

If $f \in \mathscr{L}_\phi$ then the additive functions $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\mathscr{L}_\phi$, too.

Now, for each $\delta > 0$ there exists $K = K(\delta)$ such that

$$x^{-1} \sum_{\substack{n \leqslant x \\ |\operatorname{Re} f(n)| \geqslant K}} 1 \leqslant \delta x^{-1} \sum_{n \leqslant x} \phi(|\operatorname{Re} f(n)|),$$

and the right-hand side is smaller than $\frac{1}{2}$ if δ is small enough. Then, by Lemma 1, the series

$$\sum_{\substack{p \\ |\operatorname{Re} f(p)| > 1}} \frac{1}{p} \quad \text{and} \quad \sum_{\substack{p \\ |\operatorname{Re} f(p)| \leqslant 1}} \frac{(\operatorname{Re} f(p))^2}{p}$$

converge and

$$\sum_{\substack{p \leqslant x \\ |\operatorname{Re} f(p)| \leqslant 1}} \frac{\operatorname{Re} f(p)}{p} = O(1) \quad \text{as} \quad x \to \infty.$$

The same holds for $\operatorname{Im} f$, and thus the first two series in (5) converge and (6) holds.

Now, let $p_1 < p_2 < \cdots$ denote the sequence of primes p for which $|f(p)| > 1$. Then choose an integer i_0 so large that

$$\sum_{i \geqslant i_0} p_i^{-1} < \frac{1}{30}.$$

Define an additive function $\bar{f}$ by

$$\bar{f}(p^m) = \begin{cases} f(p) & \text{if} \quad m = 1 \text{ and } |f(p)| \leqslant 1, \\ f(p) & \text{if} \quad m = 1 \text{ and } p = p_i,\ i < i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by results of Indlekofer [6], the convergence of the second series in (5) and the inequality in (6) together imply that $\sup_{x \geqslant 1} x^{-1} \sum_{n \leqslant x} |\bar{f}(n)| < \infty$ [6, Theorem 1] (and further, that $\bar{f}$ is uniformly summable [6, Theorem 4]). Therefore, there exists a K such that

$$\sup_{x \geqslant 1} x^{-1} \sum_{\substack{n \leqslant x \\ |\bar{f}(n)| \geqslant K}} 1 \leqslant \frac{1}{30}.$$

Put

$$A := \{n \colon \mu^2(n) = 1,\ |\bar{f}(n)| \leqslant K,\ p_i \nmid n \text{ for } i \geqslant i_0\}$$

and, for a given prime p,

$$A_p := \{n: n \in A,\ p \nmid n\}.$$

The number of integers $n \in A_p$ which do not exceed a given bound x is at least

$$\sum_{n \leqslant x} \mu^2(n) - \frac{1}{30} x - x \sum_{i \geqslant i_0} p_i^{-1} - xp^{-1}$$
$$\geqslant x' \left\{ 6\pi^{-2} - \frac{2}{30} - \frac{1}{2} \right\} \{1 + o(1)\} > \frac{x}{30}$$

for all $x \geqslant x_0$, say, and this holds uniformly for all primes p. Put

$$c := \max_{i \leqslant i_0} |f(p_i)|$$

and

$$B := \{p^m: 2K \leqslant |f(p^m)|\},$$

where we assume that $c < 2K$ and B is nonempty. Obviously $p_1 \notin B$ for $i \leqslant i_0$. Consider now the double sum

$$S_x := \sum_{\substack{p^m \leqslant x \\ p^m \in B}} \sum_{\substack{n \leqslant xp^{-m} \\ n \in A_p}} \phi(|f(p^m n)|).$$

We shall estimate this sum S_x, both from below and above. Observing that a typical summand satisfies

$$\phi(|f(p^m n)|) \geqslant \phi(|f(p^m)| - K) \gg \phi(|f(p^m)|),$$

we get the estimate from below

$$S_x \gg \sum_{\substack{p^m \leqslant x x_0^{-1} \\ p^m \in B}} \phi(|f(p^m)|) \sum_{\substack{n \leqslant xp^{-m} \\ n \in A_p}} 1$$
$$\geqslant \frac{x}{30} \sum_{\substack{p^m \leqslant x x_0^{-1} \\ p^m \in B}} \frac{\phi(|f(p^m)|)}{p^m}.$$

From above

$$S_x \leqslant \sum_{l \leqslant x} \phi(|f(l)|)$$

since there can be at most one representation of any integer in the form $l = p^m n$ with $p^m \in B$, $n \in A_p$.

Therefore

$$\limsup_{x \to \infty} \sum_{\substack{p^m \leq x \\ p^m \in B}} \frac{\phi(|f(p^m)|)}{p^m} \ll \limsup_{x \to \infty} x^{-1} S_x < \infty.$$

Thus, we obtain the convergence of the series

$$\sum_{2K \leq |f(p^m)|} \frac{\phi(|f(p^m)|)}{p^m}.$$

This ends the proof of Theorem 1.

5. Proof of Theorem 2

Let

$$\mathscr{P}^* := \{p: |f(p)| > 1\} \cup \{p^m: m \geq 2\}.$$

Each integer n may be uniquely decomposed into the form $n = n_1 n_2$ where n_1 contains only prime divisors of n which are not elements of $\mathscr{P}^*$, and n_2 contains the remaining primes. Then, by (7)

$$\sum_{n \leq x} \phi(|f(n)|) \ll \sum_{n_2 \leq x} \phi(|f(n_2)|) \sum_{n_1 \leq x/n_2} \phi(|f(n_1)|)$$

$$\ll \sum_{n_2 \leq x} \phi(|f(n_2)|) \sum_{n_1 \leq x/n_2} c^{|\mathrm{Re}\, f(n_1)| + |\mathrm{Im}\, f(n_1)|}. \tag{11}$$

Here we used that (7) implies $\phi(x) \ll c^x$ for some constant $c > 0$.

For the last sum in (11) we get, by using Lemma 2,

$$\sum_{n_1 \leq x/n_2} c^{|\mathrm{Re}\, f(n_1)| + |\mathrm{Im}\, f(n_1)|}$$

$$\leq \sum_{i=1} \sum_{j=1} \sum_{n_1 \leq x/n_2} c^{\varepsilon_i \mathrm{Re}\, f_1(n_1) + \varepsilon_j \mathrm{Im}\, f_1(n_1)}$$

$$\ll \frac{x}{n_2} \sum_{i=1}^{2} \sum_{j=1}^{2} \exp\left(\max\left(\sum_{p \leq x} \frac{c^{\varepsilon_i \mathrm{Re}\, f_1(p) + \varepsilon_j \mathrm{Im}\, f_1(p)} - 1}{p}\right)\right)$$

$$\ll \frac{x}{n_2},$$

where $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$.

The last inequality holds because of the relation

$$c^{\pm \operatorname{Re} f_1(p) \pm \operatorname{Im} f_1(p)} - 1 = \log c(\pm \operatorname{Re} f_1(p) \pm \operatorname{Im} f_1(p)) + O(|f_1(p)|^2),$$

the convergence of $\sum_p (|f_1(p)|^2/p)$ and (6).

Thus

$$\sum_{n \leqslant x} \phi(|f(n)|) \ll x \sum_{n_2 \leqslant x} \frac{\phi(|f(n_2)|)}{n_2}$$
$$\ll x.$$

6. Proof of Theorem 3

Because of (9) there exist sequences $\{x_\nu\}$, $\{y_\nu\}$, $x_\nu \to \infty$, $y_\nu \to \infty$ as $\nu \to \infty$, such that

$$\phi(x_\nu + y_\nu) > \nu^3 \phi(x_\nu)\,\phi(y_\nu). \tag{12}$$

Now, for every ν there are primes $p_\nu \neq p'_\nu$ so that

$$\frac{1}{2\nu \log^2 \nu} < \frac{\phi(x_\nu)}{p_\nu} < \frac{1}{\nu \log^2 \nu}$$

and

$$\frac{1}{2\nu \log^2 \nu} < \frac{\phi(y_\nu)}{p_\nu} < \frac{1}{\nu \log^2 \nu}.$$

Without loss of generality we assume that all the primes are different from each other. Denote this set of primes by $\mathscr{P}^{**}$, i.e., $\mathscr{P}^{**} = \{p_\nu, p'_\nu \mid \nu = 1, 2, 3, \ldots\}$.

We define an additive function f by

$$f(p^m) = \begin{cases} x_\nu & \text{if } \ m = 1 \text{ and } p = p_\nu \\ y_\nu & \text{if } \ m = 1 \text{ and } p = p'_\nu \\ 0 & \text{otherwise.} \end{cases}$$

Then all the series in (5) and (6) converge.

Further, we define a factorization $n = n_1 n_2$ of $n \in \mathbb{N}$. Let n_2 run through those integers which are squarefree and divisible only by primes $p \in \mathscr{P}^{**}$.

Let n_1 run through those integers n for which $p^m \| n$ then $m \geqslant 2$ or $m=1$ and $p \notin \mathscr{P}^{**}$. Let us also consider the integer 1 to be a member of the sequence $\{n_1\}$ and of the sequence $\{n_2\}$.

Since $\sum_{p \in \mathscr{P}^{**}} (1/p)$ is convergent, we obtain from elementary sieve results that $|\{n_1 < y\}| > cy$. So we have

$$\sum_{n \leqslant x} \phi(|f(n)|) = \sum_{n_1 n_2 \leqslant x} \phi(f(n_2)) \asymp x \sum_{n_2 \leqslant x} \frac{\phi(f(n_2))}{n_2}.$$

Now the last sum is bounded below by

$$\sum_{\substack{p_\nu p'_\nu \leqslant x \\ p_\nu \neq p'_\nu}} \frac{\phi(x_\nu + y_\nu)}{p_\nu p'_\nu} \geqslant \sum_{\substack{p_\nu p'_\nu \leqslant x \\ p_\nu \neq p'_\nu}} \nu^3 \frac{1}{\nu^2 \log^4 \nu},$$

and the right side tends to infinity as $x \to \infty$. This ends the proof of Theorem 3.

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